A SHORT REVIEW OF 'EIGENVECTORS FROM EIGENVALUES'

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Abstract. In this paper, I will give an example and some comparisons between the new method and the original method to calculate the eigenvectors of a Hermitian matrix.

1. Introduction

Recently, Peter B Denton, Stephen J Parke, Terence Tao, and Xining Zhang present a method of succinctly determining eigenvectors from eigenvalues in 'EIGENVECTORS FROM EIGENVALUES'. But I think this new method is more complicated than the original method. I will give a specific example to show why it is difficult to calculate.

2. An example

Let A be a $n \times n$ Hermitian matrix with eigenvalues $\lambda_i(A)$ and normed eigenvectors v_i. The elements of each eigenvector are denoted $v_{i,j}$. Let M_j be the $n-1 \times n-1$ submatrix of A that results from deleting the j^{th} column and the j^{th} th row, with eigenvalues $\lambda_k(M_j)$.

The new method showed that the norm squared of the elements of the eigenvectors are related to the eigenvalues and the submatrix eigenvalues:

$$|v_{ij}|^2 \prod_{k=1,k\neq i}^{n} (\lambda_i(A) - \lambda_k(A)) = \prod_{k=1}^{n} (\lambda_i(A) - \lambda_k(M_j))$$

We take a specific matrix and try to calculate the eigenvectors by the formula in this paper.

For example, for the Hermitian matrix $A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 2 & 3 \\ 3 & 2 & 1 & 2 \\ 0 & 3 & 2 & 1 \end{pmatrix}$

By calculating, we have $\lambda_1(A) = 2+\sqrt{26}, \lambda_2(A) = \sqrt{2}, \lambda_3(A) = -\sqrt{2}, \lambda_4(A) = 2-\sqrt{26}$

$M_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}, \lambda_1(M_1) = \frac{5+\sqrt{41}}{2}, \lambda_2(M_1) = \frac{5-\sqrt{41}}{2}, \lambda_3(M_1) = -2$

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$$M_2 = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}, \lambda_1(M_2) = 1+\sqrt{13}, \lambda_2(M_2) = 1, \lambda_3(M_2) = 1-\sqrt{13}$$

$$M_3 = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 0 & 3 & 1 \end{pmatrix}, \lambda_1(M_3) = 1+\sqrt{13}, \lambda_2(M_3) = 1, \lambda_3(M_3) = 1-\sqrt{13}$$

$$M_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}, \lambda_1(M_4) = \frac{5+\sqrt{41}}{2}, \lambda_2(M_4) = \frac{5-\sqrt{41}}{2}, \lambda_3(M_4) = -2$$

So

$$|v_{11}| = \sqrt{\frac{(\lambda_1(A)-\lambda_1(M_1))(\lambda_1(A)-\lambda_2(M_1))(\lambda_1(A)-\lambda_3(M_1))}{(\lambda_1(A)-\lambda_2(A))(\lambda_1(A)-\lambda_3(A))(\lambda_1(A)-\lambda_4(A))}} = \sqrt{\frac{26-\sqrt{26}}{104}}$$

Simlarly, we can get $|v_{12}| = \sqrt{\frac{26+\sqrt{26}}{104}}$, $|v_{13}| = \sqrt{\frac{26+\sqrt{26}}{104}}$, $|v_{14}| = \sqrt{\frac{26-\sqrt{26}}{104}}$,
$|v_{21}| = \sqrt{\frac{26+13\sqrt{2}}{104}}$, $|v_{21}| = \sqrt{\frac{26-13\sqrt{2}}{104}}$, $|v_{23}| = \sqrt{\frac{26-13\sqrt{2}}{104}}$, $|v_{24}| = \sqrt{\frac{26+13\sqrt{2}}{104}}$,
$|v_{31}| = \sqrt{\frac{26+13\sqrt{2}}{104}}$, $|v_{32}| = \sqrt{\frac{26-13\sqrt{2}}{104}}$, $|v_{33}| = \sqrt{\frac{26-13\sqrt{2}}{104}}$, $|v_{34}| = \sqrt{\frac{26+13\sqrt{2}}{104}}$,
$|v_{41}| = \sqrt{\frac{26+\sqrt{26}}{104}}$, $|v_{42}| = \sqrt{\frac{26-\sqrt{26}}{104}}$, $|v_{43}| = \sqrt{\frac{26-\sqrt{26}}{104}}$, $|v_{44}| = \sqrt{\frac{26+\sqrt{26}}{104}}$.

Here, A a real symmetric matrix. So by "the eigenvectors of the different eigenvalues of the real symmetric matrix can be all real vectors" and "the eigenvectors of the different eigenvalues of the Hermite matrix are orthogonal", we can know whether v_{ij} is positive or negative.

By testing the sign for -1 and 1, we finally know that

$$v_1 = \left(-\sqrt{\frac{26+\sqrt{26}}{104}}, -\sqrt{\frac{26+\sqrt{26}}{104}}, -\sqrt{\frac{26+\sqrt{26}}{104}}, -\sqrt{\frac{26-\sqrt{26}}{104}}\right),$$

$$v_2 = \left(-\sqrt{\frac{26+13\sqrt{2}}{104}}, \sqrt{\frac{26-13\sqrt{2}}{104}}, -\sqrt{\frac{26-13\sqrt{2}}{104}}, \sqrt{\frac{26+13\sqrt{2}}{104}}\right),$$

$$v_3 = \left(\sqrt{\frac{26+13\sqrt{2}}{104}}, \sqrt{\frac{26-13\sqrt{2}}{104}}, -\sqrt{\frac{26-13\sqrt{2}}{104}}, -\sqrt{\frac{26+13\sqrt{2}}{104}}\right),$$

$$v_4 = \left(-\sqrt{\frac{26+\sqrt{26}}{104}}, \sqrt{\frac{26-\sqrt{26}}{104}}, \sqrt{\frac{26-\sqrt{26}}{104}}, -\sqrt{\frac{26+\sqrt{26}}{104}}\right).$$

3. Conclusion

In "EIGENVECTORS FROM EIGENVALUES", they wrote that"The form of Lemma 2 with the norm squared of the elements of the eigenvectors is expected in that any determination of the eigenvectors from the eigenvalues is insensitive to the phases since one can multiply any eigenvector by a phase $e^{i\theta}$ while leaving A, M_j , and the eigenvalues unchanged."

For the real symmetric matrix, just like the example, we can know whether v_{ij} is positive or negative by testing. However, for the general Hermitian matrix, the phases lack a good way to determine.

If you use this new method to calculate the eigenvectors of a real symmetric matrix, then you need to follow these steps:

1. Calculate n eigenvalues of the original matrix. (the time complexity is $O\left(n^3\right)$)

2. Calculate $n-1$ eigenvalues of n sub-matrix. (the time complexity is $O\left(n^4\right)$)
3. Determine the sign (the time complexity is more than $O\left(n^3\right)$)

However, the traditional method only requires two steps:

1. Find its corresponding eigenvalue (the time complexity is $O\left(n^3\right)$);
2. Solve the homogeneous linear equations (the time complexity is $O\left(n^3\right)$).

What's more, Tao wrote in his blog: numerical problems will arise if the Hermitian matrix (or a particular submatrix of it) has coincident (or nearly-coincident) eigenvalues.

So all in all, the new method has a huge amount of calculations and is limited in many aspects.

References

[1] Peter B Denton, Stephen J Parke, Terence Tao, Xining Zhang, EIGENVECTORS FROM EIGENVALUES

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